Math 106 – Notes
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Stochastic integration: Motivation

In the "physics" view, the Wiener process is interpreted as the time integral of a non-existent process called white noise $\dot{W}_t$. Although the time derivative of a Wiener process does not exist, it is still useful to regard $\dot{W}_t$ heuristically as a stationary Gaussian process. This viewpoint can be motivated by examining finite–difference approximations to the derivative of a Wiener process over a time interval $h > 0$. Observe that

$$\mathbb{E}\left[\frac{W_{t+h} - W_t}{h}\right] = 0, \tag{1}$$

$$\mathbb{E}\left[\left(\frac{W_{t+h} - W_t}{h}\right)^2\right] = \frac{1}{h}, \tag{2}$$

$$\mathbb{E}\left[\left(\frac{W_{t+h} - W_t}{h}\right)\left(\frac{W_{s+h} - W_s}{h}\right)\right] = 0 \qquad \text{for } t + h < s. \tag{3}$$

Thus the increments behave like independent Gaussian variables with variance $1/h$, and in the limit $h \to 0$ this suggests a covariance kernel

$$K(s, t) = \delta(t - s),$$

which is the kernel of idealized white noise.

White noise provides a natural way to construct continuous-state, continuous-time Markov processes by adding uncorrelated perturbations to an ODE:

$$\frac{d}{dt} X_t = b(X_t) + \sigma(X_t)\, \dot{W}_t. \tag{4}$$

This informal equation says that the instantaneous velocity is perturbed by a Gaussian fluctuation of magnitude $\sigma(X_t)$, independent across time. The stochastic differential equation (4) defines a diffusion process. Special cases include the Wiener process ($b = 0$, $\sigma = 1$) and the Ornstein–Uhlenbeck process ($b(x) = -\gamma x$, $\sigma = \sqrt{2D}$).

Because paths of X_t are nowhere differentiable, the differential form (4) must be interpreted as an integral equation:

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, \dot{W}_s\, ds. \tag{5}$$

The second term is interpreted as a stochastic integral. Since

$$\int_0^t \sigma(X_s)\, \dot{W}_s\, ds = \int_0^t \sigma(X_s)\, dW_s,$$

we must define $\int_0^t f(\omega, s)\, dW_s$ in a mathematically meaningful way where $f(\omega, s)$ is some adapted function. We want to make the approximation

$$\int_0^t f(\omega, s)\, dW_s \approx \frac{1}{N} \sum_{i=1}^{N} f(\omega, t^*)(W_{t_{i+1}} - W_{t_i}) \tag{6}$$

The usual way of defining the Riemann–Stieltjes where the limit is independent of where t^* falls does not hold, as we now show.

Three definitions of stochastic integrals

Consider the example where $f(\omega, s) = W_s(\omega)$. Thus the problem is to define

$$I = \int_0^t W_s\, dW_s. \tag{7}$$

Let $0 = t_0 < t_1 < \cdots < t_N = t$ be a partition with mesh $|\Pi_N| \to 0$.

Left-endpoint (Itô) sum.

$$I_N^L = \sum_{j=0}^{N-1} W_{t_j} \left(W_{t_{j+1}} - W_{t_j}\right). \tag{8}$$

Because W_{t_j} is measurable with respect to $\mathcal{F}_{t_j}$ and increments are independent with mean zero, $\mathbb{E}[I_N^L] = 0$.

Right-endpoint.

$$I_N^R = \sum_{j=0}^{N-1} W_{t_{j+1}} \left(W_{t_{j+1}} - W_{t_j}\right). \tag{9}$$

Rewrite $W_{t_{j+1}}$ as $W_{t_j} + (W_{t_{j+1}} - W_{t_j})$:

$$I_N^R = \sum_{j=0}^{N-1} \left(W_{t_j} + (W_{t_{j+1}} - W_{t_j})\right)\left(W_{t_{j+1}} - W_{t_j}\right). \tag{10}$$

This expands to

$$I_N^R = \sum_{j=0}^{N-1} W_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right) + \sum_{j=0}^{N-1} (W_{t_{j+1}} - W_{t_j})^2. \tag{11}$$

Hence

$$I_N^R = I_N^L + \sum_{j=0}^{N-1} (W_{t_{j+1}} - W_{t_j})^2. \tag{12}$$

Taking expectations,

$$\mathbb{E}[I_N^R] = \mathbb{E}[I_N^L] + \sum_{j=0}^{N-1} \mathbb{E}[(W_{t_{j+1}} - W_{t_j})^2] = 0 + \sum_{j=0}^{N-1} (t_{j+1} - t_j) = t. \tag{13}$$

Midpoint (Stratonovich) sum. Let $t_{j+\frac{1}{2}} = \frac{t_j + t_{j+1}}{2}$. The midpoint Riemann sum is

$$I_N^M = \sum_{j=0}^{N-1} W_{t_{j+\frac{1}{2}}} \left(W_{t_{j+1}} - W_{t_j}\right). \tag{14}$$

Write the increment as

$$W_{t_{j+1}} - W_{t_j} = \left(W_{t_{j+1}} - W_{t_{j+\frac{1}{2}}}\right) + \left(W_{t_{j+\frac{1}{2}}} - W_{t_j}\right). \tag{15}$$

Then

$$\mathbb{E}[I_N^M] = \sum_{j=0}^{N-1} \mathbb{E}\Big[W_{t_{j+\frac{1}{2}}} \left(W_{t_{j+1}} - W_{t_{j+\frac{1}{2}}}\right)\Big] + \sum_{j=0}^{N-1} \mathbb{E}\Big[W_{t_{j+\frac{1}{2}}} \left(W_{t_{j+\frac{1}{2}}} - W_{t_j}\right)\Big]. \tag{16}$$

The first sum is zero and the second sum is $(t_{j+1} - t_j)/2$ by the same argument we used for the right hand rule.

$$\mathbb{E}[I_N^M] = \sum_{j=0}^{N-1} \frac{t_{j+1} - t_j}{2} = \frac{t}{2}. \tag{17}$$

The Itô Integral

There are both physical and mathematical reasons to prefer the left-endpoint (Itô) definition.

- Mathematically, using the left endpoint ensures that the stochastic integral depends only on information available up to the current time.
- Physically, many stochastic models arise as limits of discrete-time systems in which updates are computed using only the current state before the noise is applied. Taking a continuous-time limit of such adapted, forward-looking update rules naturally leads to the left-endpoint convention. In this sense, the Itô integral directly reflects how real noisy systems are typically constructed from their discrete approximations.

Thus, we shall focus on defining the Itô integral and it should general be assumed that

$$\int_0^t f(\omega, s) dW_s \tag{18}$$

is referring to the Itô Integral.

Itô isometry

To rigorously define the Itô integral, we want to show that the map

$$I(f) : f \ \mapsto \ \int_0^T f \, dW \tag{19}$$

is a well-defined. This mapping is a linear operator from the space

$$\mathcal{V} = \Big\{ f(\omega, t) : f \text{ is } \mathcal{F}_t^W\text{-adapted and } \mathbb{E}\Big[\int_0^T f(\omega, t)^2 \, dt\Big] < \infty \Big\}. \tag{20}$$

of square-integrable adapted processes to $L_2(\Omega)$. The result that establishes the validity of the Itô integral is called the Itô isometry because it establishes an Isometry of the Hilbert spaces $\mathcal{V}$ and $L_2(\Omega)$. Remember an isometry is a distance preserving map and the distance on a Hilbert space is induced by the inner product.

Notation: We can also write

$$\mathcal{V} = L^2(\Omega \times [0,T]) \cap \{f \text{ is } \mathcal{F}_t^W\text{-adapted}\} \tag{21}$$

where $L^2(\Omega \times [0,T])$ is sometimes denoted $L^2_\omega L^2_t$. Think of it just like the L^2 space of real values functions on $[0,T]$, except that now we are considering functions on the larger space $\Omega \times [0,T]$. The expectation therefore plays the role of an outer integral here. In the book they use the interval $[S,T]$ but I'll set $S = 0$ to avoid the extra notation.

Simple functions

We begin with *simple* integrands of the form

$$f(\omega,t) = \sum_{j=1}^{n} e_j(\omega)\, \mathbf{1}_{[t_j,t_{j+1})}(t),$$

where each e_j is $\mathcal{F}_{t_j}$–measurable random variable. I'll use $\mathcal{S} \subset \mathcal{V}$ to denote the space of such functions.

We assume throughout that $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmented filtration generated by the Wiener process, so W_t is adapted and e_j depends only on information available at time t_j.

For such simple functions, the Itô integral on $[0,T]$ is defined by the left-endpoint rule:

$$\int_0^T f(\omega,t)\, dW_t = \sum_j e_j(\omega)\left(W_{t_{j+1}} - W_{t_j}\right).$$

Lemma 1 (T7.1)**.** *Let $0 \leq S \leq T$. For a simple integrand f as above, the stochastic integral satisfies*

$$\mathbb{E}\left[\int_S^T f(\omega,t)\, dW_t\right] = 0, \tag{22}$$

and the Itô isometry

$$\mathbb{E}\left[\left(\int_S^T f(\omega,t)\, dW_t\right)^2\right] = \mathbb{E}\left[\int_S^T f^2(\omega,t)\, dt\right] \tag{23}$$

Proof. We set $S = 0$ throughout. For the second moment, let

$$\delta W_j = W_{t_{j+1}} - W_{t_j}.$$

Then

$$\mathbb{E}\left[\left(\sum_j e_j\, \delta W_j\right)^2\right] = \mathbb{E}\left[\sum_{j,k} e_j e_k\, \delta W_j\, \delta W_k\right] = \sum_{j,k} \mathbb{E}\big[e_j e_k\, \delta W_j\, \delta W_k\big].$$

By independence of the increments, δW_j and δW_k are independent for $j \neq k$ and have mean zero, so the off–diagonal terms vanish and

$$\mathbb{E}\left[\left(\sum_j e_j\, \delta W_j\right)^2\right] = \sum_j \mathbb{E}\left[e_j^2\, (\delta W_j)^2\right] = \sum_j \mathbb{E}\left[e_j^2\right](t_{j+1} - t_j),$$

since $\mathbb{E}[(\delta W_j)^2] = t_{j+1} - t_j$. For this simple process,

$$\int_0^T f^2(\omega, t)\, dt = \sum_j e_j(\omega, t_j)^2\, (t_{j+1} - t_j),$$

to taking expectations gives the result $\square$

Extension to $\mathcal{V}$

Now we want to extend this to any $f \in \mathcal{V}$. If $f \in \mathcal{V}$, then there exists a sequence $\{\varphi_n\} \subset \mathcal{S}$ such that

$$\mathbb{E}\left[\int_0^T \big(f(\omega, t) - \varphi_n(\omega, t)\big)^2 dt\right] \longrightarrow 0.$$

That is, $\varphi_n \to f$ in the space $L^2(\Omega \times [0, T])$. For simple functions, the Itô integral has already been defined. We therefore set

$$\int_0^T f(\omega, t)\, dW_t = \lim_{n\to\infty} \int_0^T \varphi_n(\omega, t)\, dW_t, \tag{24}$$

where the limit is taken in $L^2(\Omega)$. Note how we are going from convergence in $L^2(\Omega \times [0, T])$ to $L^2(\Omega)$.

As an intermediate step towards proving the general isometry, we need the following facts.

Lemma 2. *1. The limit in Eq. 24 is well-defined in the sense of belonging to $L^2(\Omega)$ and*

2. the limiting value does not depend on the choice of $\{\varphi_n\}$ which are used to approximate f (this is E7.1)

Proof. 1. Note that $\int_0^T \varphi_n(\omega, t)\, dW_t \in L^2(\Omega)$ by the Itô isometry for simple functions. Furthermore, for any n, m,

$$\mathbb{E}\left[\left(\int_0^T \varphi_n\, dW_t - \int_0^T \varphi_m\, dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T (\varphi_n - \varphi_m)^2\, dt\right].$$

Because $\{\varphi_n\}$ is a Cauchy sequence in $L^2(\Omega \times [0, T])$, the right-hand side tends to zero. Hence the sequence

$$\left\{\int_0^T \varphi_n\, dW_t\right\}$$

is Cauchy in $L^2(\Omega)$, and therefore converges to a unique limit.

2. Let $f \in \mathcal{V}$. Suppose $\{\varphi_n\}$ and $\{\psi_n\}$ are two sequences which both converge to f in $\mathcal{V}$. Define

$$X_n := \int_0^T \varphi_n(\omega, t)\, dW_t, \qquad Y_n := \int_0^T \psi_n(\omega, t)\, dW_t.$$

We already know from the Itô isometry that both $\{X_n\}$ and $\{Y_n\}$ are Cauchy sequences in $L^2(\Omega)$, hence converge in $L^2(\Omega)$ to some limits X and Y, respectively. We must show that $X = Y$ a.s. The idea is again to use the Itô isometry for simple functions to leverage the convergence in $\mathcal{V}$.

$\square$

We can now prove the general Itô isometry, hence establishing the validity of the Itô integral.

Theorem 1 (T7.2)**.** *T7.1 holds for all $f \in \mathcal{V}$.*

Proof. I'll leave Eq. 22 as an exercise. So far we've proved that the Itô Integral viewed as a map $I : L^2(\Omega \times [0,T]) \to L^2(\Omega)$ is well defined and can be computed by approximating $f \in \mathcal{V}$ with simple functions. It remains to show that it is an isometry. For this, we use the following fact: If $X_n \to X$ in some Hilbert space H, then $||X_n|| \to ||X||$. We have already seen that

$$\int_0^t \varphi_n(\omega,s)dW_s \to \int_0^t f(\omega,s)dW_s \tag{25}$$

in $L^2(\Omega \times [0,T])$ and $\varphi_n \to f(\omega,s)$ in $L^2(\Omega)$. The Itô isometry for the limits follows from the Itô isometry for simple functions along with the fact that the norms (and hence their squares) converge as well. $\square$

Properties of the Itô integral

Proposition 1. *Let $f, g \in \mathcal{V}$ and let $u \in [0,T]$. Then the Itô integral satisfies:*

(i)

$$\int_0^T f\,dW_t = \int_0^u f\,dW_t + \int_u^T f\,dW_t$$

(ii) For any constant $c \in \mathbb{R}$,

$$\int_0^T (f + cg)\,dW_t = \int_0^T f\,dW_t + c\int_0^T g\,dW_t,$$

(iii) The random variable

$$\int_0^T f\,dW_t$$

is $\mathcal{F}_T$–measurable.

Example (E7.5) and Quadratic variation

Now that we've established the Itô integral exists, let's revisit the integral I defined by Eq. 7. We now compute the Itô integral $\int_0^t W_s\,dW_s$ using left-endpoint Riemann sums. Let $\{t_j\}$ be a partition of $[0,t]$. Then

$$\int_0^t W_s\,dW_s \;\approx\; \sum_j W_{t_j}\,(W_{t_{j+1}} - W_{t_j}).$$

Using the identity

$$2ab = (a^2 - b^2) - (a-b)^2,$$

we rewrite each summand:

$$W_{t_j}(W_{t_{j+1}} - W_{t_j}) = \frac{2W_{t_j}W_{t_{j+1}} - 2W_{t_j}^2}{2} = \frac{W_{t_{j+1}}^2 - W_{t_j}^2}{2} - \frac{(W_{t_{j+1}} - W_{t_j})^2}{2}.$$

Summing over j gives

$$\sum_j W_{t_j}(W_{t_{j+1}} - W_{t_j}) = \sum_j \frac{W_{t_{j+1}}^2 - W_{t_j}^2}{2} - \frac{1}{2}\sum_j (W_{t_{j+1}} - W_{t_j})^2.$$

The first sum telescopes:

$$\sum_j \frac{W_{t_{j+1}}^2 - W_{t_j}^2}{2} = \frac{W_t^2}{2}.$$

The second sum is related to the *quadratic variation*, defined as

$$[W, W]_t = \lim_{dt \to 0} \sum_j (W_{t_{j+1}} - W_{t_j})^2 \tag{26}$$

It can be shown that $[W_t, W_t] = t$. To see why, consider that (as in Eq. 2)

$$\mathbb{E}[(W_{t_{j+1}} - W_{t_j})^2] = dt \tag{27}$$

To connect this back to the initial discussion of White noise, we can formally write

$$(\dot{W}_t)^2 = \left(\frac{dW_t}{dt}\right)^2 \approx \frac{1}{dt} \tag{28}$$

Putting things together yields

$$\int_0^t W_s \, dW_s = \frac{W_t^2}{2} - \frac{t}{2}.$$

Generalizing Eq. 26, it can be shown that (see P7.6)

$$\sum_j f(\omega, t_j^*)(W_{t_{j+1}} - W_{t_j})^2 \to \int_0^T f(\omega, t) dt. \tag{29}$$

where $t_j^* \in [t_j, t_{j+1}]$ is arbitrary. For this reason, we sometimes write $(dW_t)^2 = dt$.

Itô's formula

If we interpret the example above in differential form this says

$$\frac{d}{dt} W_t^2 = 2W_t \dot{W}_t + 1 \tag{30}$$

This illustrates a fundamental difference between standard calculus and so-called Itô calculus: The regular chain rule does not hold. In differential form we would write this as

$$d(W_t)^2 = 2W_t dW_t + dt \tag{31}$$

This generalizes to the central result of stochastic integration theory. To state this, we consider a process X_t given by Eq. 32; that is,

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s. \tag{32}$$

We call this an *Itô process*. The functions b and σ could also depend on time, but I will leave it as an exercise to check that nothing essential changes. We want to understand how such an expression behaves under a change of variables.

It is useful to express Eq. 32 in differential form as

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t. \tag{33}$$

Suppose we want to compute $df(X_t)$ for a twice differentiable function f. A formal Taylor expansion gives

$$df(X_t) \approx f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, (dX_t)^2 \tag{34}$$

To make sense of this expansion, we need to understand the size of the term $(dX_t)^2$. From Eq. 33,

$$(dX_t)^2 = b(X_t)^2 (dt)^2 + 2b(X_t)\sigma(X_t)\, dt\, dW_t + \sigma(X_t)^2 (dW_t)^2. \tag{35}$$

We compare each of these terms to dt:

- The term $(dt)^2$ is negligible, since $(dt)^2 = o(dt)$.
- The mixed term $dt\, dW_t$ is also negligible: because $dW_t = O(\sqrt{dt})$.
- We've already seen $(dW_t)^2 = dt$.

Keeping only terms of order dt and discarding terms of smaller order, Eq. 35 reduces to

$$(dX_t)^2 = \sigma(X_t)^2\, dt. \tag{36}$$

Substituting this into the Taylor expansion (34) leads directly to

$$df(X_t) = f'(X_t) dX_t + f''(X_t)\sigma(X_t)^2 dt. \tag{37}$$

Note again the contrast with the usual Chain rule:

$$\frac{d}{dt} f(X_t) = f'(X_t) \frac{d}{dt} X_t + f''(X_t)\sigma(X_t)^2 \tag{38}$$

Theorem 2 (Itô's formula in one dimension T7.7)**.** *Define $Y_t = f(X_t)$. Then Y_t is also an Itô process, and its differential is*

$$dY_t = \left(b(X_t, t)\, f'(X_t) + \frac{1}{2}\, \sigma^2(X_t, t)\, f''(X_t) \right) dt \;+\; \sigma(X_t, t)\, f'(X_t)\, dW_t.$$

The idea of the proof is to approximate b and σ using simple functions.

This generalizes to multidimensional processes

Theorem 3 (Multidimensional Itô formula T7.9)**.** *Let $\mathbf{X}_t \in \mathbb{R}^n$ be a process of the form*

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t \mathbf{b}(\mathbf{X}_s)\, ds + \int_0^t \boldsymbol{\sigma}(\mathbf{X}_s)\, d\mathbf{W}_s,$$

where $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{\sigma} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\mathbf{W}_t \in \mathbb{R}^m$ is an m–dimensional Wiener process. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable and set $Y_t = f(\mathbf{X}_t)$. Then Y_t satisfies

$$dY_t = \left(\nabla f(\mathbf{X}_t)^T \mathbf{b}(\mathbf{X}_t) + \frac{1}{2}\, \boldsymbol{\sigma}(\mathbf{X}_t)\boldsymbol{\sigma}(\mathbf{X}_t)^T : \nabla^2 f(\mathbf{X}_t) \right) dt \;+\; \nabla f(\mathbf{X}_t)^T \boldsymbol{\sigma}(\mathbf{X}_t)\, d\mathbf{W}_t. \tag{39}$$

In component form, write the SDE as

$$dX_t^i = b_i(X_t)\,dt + \sum_{k=1}^{m} \sigma_{ik}(X_t)\,dW_t^k, \qquad i = 1, \dots, n. \tag{40}$$

Then Itô's formula becomes

$$\begin{aligned} dY_t = \sum_{i=1}^{n} f_{x_i}(X_t)\,b_i(X_t)\,dt + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{m}\sigma_{ik}(X_t)\,\sigma_{jk}(X_t)\right) f_{x_i x_j}(X_t)\,dt \\ + \sum_{k=1}^{m}\left(\sum_{i=1}^{n} f_{x_i}(X_t)\,\sigma_{ik}(X_t)\right) dW_t^k. \end{aligned} \tag{41}$$

Examples

Consider the Ornstein–Uhlenbeck SDE

$$dX_t = -\gamma X_t\,dt + \sigma\,dW_t. \tag{42}$$

To solve it using an integrating factor, introduce a second process Z_t defined by

$$dZ_t = \gamma Z_t\,dt, \qquad Z_0 = 1, \tag{43}$$

so that $Z_t = e^{\gamma t}$. We view (Z_t, X_t) as a two–dimensional process for which the drift vector and diffusion matrix are

$$\mathbf{b}(z,x) = \begin{pmatrix} \gamma z \\ -\gamma x \end{pmatrix}, \quad \boldsymbol{\sigma}(z,x) = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}$$

Let $Y_t = f(Z_t, X_t) = Z_t X_t$. Since all second derivatives of f vanish, the Hessian term in the Itô formula is zero. For the drift term,

$$\nabla f(Z_t, X_t)^T \mathbf{b}(Z_t, X_t) = \begin{pmatrix} X_t \\ Z_t \end{pmatrix}^T \begin{pmatrix} \gamma Z_t \\ -\gamma X_t \end{pmatrix} = \gamma Z_t X_t - \gamma Z_t X_t = 0.$$

For the diffusion term,

$$\nabla f(Z_t, X_t)^T \boldsymbol{\sigma}(Z_t, X_t) = \begin{pmatrix} X_t \\ Z_t \end{pmatrix}^T \begin{pmatrix} 0 \\ \sigma \end{pmatrix} = \sigma Z_t.$$

Therefore,

$$d(Z_t X_t) = \sigma Z_t\,dW_t. \tag{44}$$

Integrating from 0 to t gives

$$Z_t X_t - Z_0 X_0 = \sigma \int_0^t Z_s\,dW_s.$$

Since $Z_0 = 1$ and $Z_t = e^{\gamma t}$, this becomes

$$e^{\gamma t} X_t = X_0 + \sigma \int_0^t e^{\gamma s}\,dW_s.$$

Multiplying both sides by $e^{-\gamma t}$ yields the explicit OU solution:

$$X_t = e^{-\gamma t} X_0 + \sigma \int_0^t e^{-\gamma(t-s)}\, dW_s. \tag{45}$$

In this case, we get the same thing as we would get applying ODE methods and treating the dW_s term as time a determinstic dependent component. This gives the classical representation of the Ornstein–Uhlenbeck process as an exponentially damped initial condition plus a stochastic convolution with the Wiener process.

Geometric Brownian Motion

Consider the stochastic differential equation

$$dX_t = rX_t\, dt + \alpha X_t\, dW_t, \qquad X_0 > 0, \tag{46}$$

which defines a geometric Brownian motion. To solve it, we apply Itô's formula to $f(x) = \log x$. Since $X_t > 0$ almost surely, this is well defined. Itô's formula gives

$$\begin{aligned} d(\log X_t) &= \left(\frac{1}{X_t}\, rX_t + \frac{1}{2}\left(-\frac{1}{X_t^2} \right) \alpha^2 X_t^2 \right) dt + \frac{1}{X_t}\, \alpha X_t\, dW_t \\ &= \left(r - \tfrac{1}{2}\alpha^2\right) dt + \alpha\, dW_t. \end{aligned}$$

Thus we obtain the linear SDE

$$d(\log X_t) = \left(r - \tfrac{1}{2}\alpha^2\right) dt + \alpha\, dW_t. \tag{47}$$

Integrate both sides from 0 to t:

$$\log X_t - \log X_0 = \left(r - \tfrac{1}{2}\alpha^2\right) t + \alpha W_t.$$

Exponentiating gives the explicit solution

$$X_t = X_0 \exp\left[\left(r - \tfrac{1}{2}\alpha^2\right) t + \alpha W_t\right]. \tag{48}$$

This is the standard closed form for geometric Brownian motion, which underlies the Black–Scholes model and many stochastic growth models.