Math 106 – Midterm 1 Practice Problems

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The actual midterm will require you to solve 3 to 4 problems out of 4 to 6. The exam will cover the material from Chapters 3 and 5 which was covered in class and the readings. The problems below include, in my opinion, a range of difficulty levels and so will the exam.

Discrete-time Markov Chains and Q-processes (Chapter 3)

1. Suppose that a transition matrix of a discrete Markov chain on $\{1,\dots,2K\}$ can be written the form
$$P = \begin{bmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{bmatrix}. \tag{1}$$
where A and B are $K \times K$ matrices with all positive elements and $\mathbf{0}$ is a zero matrix. Is P primitive? Prove or disprove.

2. Consider the discrete time Markov chain $\{Y_n\}_{n\in\mathbb{N}}$ with transition matrix
$$P = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 1/2 & 1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2}$$
Let $\{H_n\}_{n\geq 1}$ be iid exponentially distributed times with rate $\lambda = 1$. Define
$$K_t = \sup\left\{ n \geq 0 : \sum_{i=1}^{n} H_i \leq t \right\} \tag{3}$$
$$X_t = Y_{K_t} \tag{4}$$
Then $\{X_t\}_{t\geq 0}$ is a Q-process. What is the Q matrix?

3. Let $\{X_t\}_{t\geq 0}$ be the Q-process on $\{1,2,3\}$ with matrix
$$Q = \begin{bmatrix} -6 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \tag{5}$$
and assume $X_0 = 1$. Let
$$T_2 = \inf\{t \geq 0 : X_t = 2\} \tag{6}$$
Find the distribution or probability density function of T_2.

4. Consider the Q-process where $Q_{i,j} = 1$ for $i \neq j$. Prove that Q has an equilibrium steady-state (i.e. the time reversed processes is indistinguishable from the forward process).

5. Write down the Q matrix for a Q-process on $\{1,2,3\}$ for which the steady-state distribution is
$$\pi = (1/2, 1/4, 1/4). \tag{7}$$
Hint: use detailed balance.

6. The δ-skeleton a Q-process is the discrete-time process $\{W_n\}_n$ with $W_n = X_{\delta n}$ $(\delta > 0)$. Prove that the δ-skeleton of an irreducible Q-process is always primitive.

7. Let N_t be a Poisson process with rate λ. Compute $\mathrm{cov}(N_t, N_s)$.

8. Let N_t be a Poisson process and $Y_t = N_t + X$ where X is a random variable on $\mathbb{N}$ write down the backward equation for Y_t; that is, the differential equations equation for $h_n(t) = \mathbb{E}[f(Y_t)|Y_0 = 0]$. Include the initial conditions.

9. Let $\Pi_1(t)$ and $\Pi_2(t)$ be independent unit rate ($\lambda = 1$) Poisson process. Consider
$$X_t = \Pi_1(\alpha t) - \Pi_2(\beta t), \quad \alpha, \beta > 0 \tag{8}$$
Write down the forward equation; that is, the differential equation for $p_n(t) = \mathbb{P}(X_t = n)$ of X_t.

10. Using the same notation as the previously problem, consider the process X_t satisfying
$$X_t = \Pi_1(\alpha t) - \Pi_2\left(\beta \int_0^t X_s ds\right) \tag{9}$$
Write down the forward equation.

Filtration and stopping times (Chapter 5)

11. Prove that single points on $\mathbb{R}$ are in $\mathcal{R}$, the Borel σ-algebra.

12. For a Poisson process, describe the filtration. In particular, what is the natural path space Ω and what sets generate the filtration?

13. Consider the filtration $\{\mathcal{F}_n\}_{n>0}$ associated with the subset of sequence space $\Omega \subset \{1,2,3\}^{\mathbb{N}}$ whose first three elements are
$$\begin{aligned}\mathcal{F}_0 &= \sigma(C(1), C(2))\\ \mathcal{F}_1 &= \sigma(C(1,2), C(2,3))\\ \mathcal{F}_2 &= \sigma(C(1,2,3), C(2,3,1))\end{aligned}$$
where C denotes the cylinder sets
$$C(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) = \{\omega \in \Omega : \omega_0 = \varepsilon_0, \dots, \omega_n = \varepsilon_n\}. \tag{10}$$
Let X_n be an $\{\mathcal{F}_n\}_{n>0}$-adapted time-homogenous Markov chain.
    - What can you say about this process? In particular, provide as much information as possible about its initial conditions and entries of the transition matrix.
    - What is $\mathcal{F}_3$?

14. Consider a discrete-time Markov Chain $\{X_n\}_{n\geq 0}$ on $\Omega_0 = \{1, 2, 3\}$ with $X_0 = 1$. Given a description of $\mathcal{F}_2^X$ either explicitly or in terms of the sets that generate it.

15. Let $\{X_t\}_{t\geq 0}$ be a continuous time stochastic process and suppose T_1 and T_2 are stopping times.
    - Prove that $T_1 + T_2$ is a stopping time?
    - Is the event $\{T_1 > T_2\}$ in $\mathcal{F}_t^X$ for fixed t?

Gaussian Processes (Chapter 5)

16. Let X_t and Y_t be mean zero Gaussian processes with kernels $K_1(t, s)$ and $K_2(t, s)$. Show that $X_t + Y_t$ is a Gaussian processes and derive its kernel.

17. Let X_t be a Gaussian process and assume $Y_t = dX_t/dt$ is also a GP. Derive the kernel function of Y_t.

18. Consider the kernel

$$K(t, s) = Ae^{-\sin(\pi|t-s|)^2} \tag{11}$$

Let X_t be a Gaussian process with this kernel and sketch some realizations of X_t. Hint: Think about where X_t and X_s are perfectly correlated.