Math 106 – Midterm Exam

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Instructions.

- You must solve **4 out of the 7 problems**. Clearly mark which problems you would like me to grade. You may attempt as many as you like, but I will only grade 4.
- You must solve **2 problems from each section**.
- No notes, no electronics.

Each problem is weighted equally.

Section 1

1. Consider the transition Matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

Determine whether P is (i) irreducible and (ii) primitive.

Solution. (i) For any two states, we can find a path both ways, which is easy to see from the graph. (ii) Observe that if it is in states $\Omega_1 = \{2, 3\}$ as time n then it is in states $\Omega_2 = \{1, 4\}$ at time $n + 1$ and vice versa. Thus for odd steps, we the transition rates within subsets Ω_1 and Ω_2 are zero, implying P is not primitive. Alternatively, reorder the states via the permutation $\sigma = (\sigma(1), \sigma(2), \sigma(3), \sigma(4)) = (1, 4, 2, 3)$. Under the new labeling the matrix has the form

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix},$$

with A, B strictly positive matrices. This is the same as the matrix in the midterm practice problem 1, which can be shown to be primitive by induction.

2. Let $\{X_t\}_{t\geq 0}$ be a Q–process on $\{1, 2, 3\}$ with generator

$$Q = \begin{bmatrix} -3 & 1 & 2 \\ 4 & -7 & 3 \\ 0 & 5 & -5 \end{bmatrix},$$

and $X_0 = 1$. Let $\{Y_n\}_{n\geq 0}$ be the embedded chain and $\{H_n\}_{n>0}$ be the holding times. Compute the following

- $\mathbb{P}(Y_2 = 3)$
- $\mathbb{P}(H_2 < 1)$

Solution. The only way to have $Y_2 = 3$ starting from $Y_0 = 1$ is $1 \to 2 \to 3$, since from 3 the next jump must go to 2. Hence

$$\mathbb{P}(Y_2 = 3) = \mathbb{P}(Y_1 = 2)\,\mathbb{P}(Y_2 = 3 \mid Y_1 = 2) = \frac{1}{3} \cdot \frac{3}{7} = \frac{1}{7}.$$

The holding time H_2 is exponential with rate $-q_{Y_1Y_1}$, conditional on Y_1. We have

$$\mathbb{P}(Y_1 = 2) = \tfrac{1}{3}, \quad \mathbb{P}(Y_1 = 3) = \tfrac{2}{3},$$

and the rates are 7 from state 2 and 5 from state 3. Therefore

$$\mathbb{P}(H_2 < 1) = \frac{1}{3}(1 - e^{-7}) + \frac{2}{3}(1 - e^{-5}).$$

3. Let $\{Y_n\}_{n\geq 0}$ be a discrete–time Markov chain on $\{1, 2, 3\}$ with transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \end{bmatrix}.$$

Define a sequence of holding times $\{H_n\}_{n\geq 1}$ such that

$$H_n \mid (Y_{n-1} = 1) \sim \text{Exponential}(1), \qquad H_n \mid (Y_{n-1} \neq 1) \sim \text{Exponential}(2),$$

Set

$$K_t = \sup\left\{ n \geq 0 : \sum_{k=1}^{n} H_k \leq t \right\}, \qquad X_t = Y_{K_t}.$$

Find the Q matrix.

Solution. When $X_t = i$, jumps are attempted at rate

$$\lambda(1) = 1, \qquad \lambda(2) = \lambda(3) = 2.$$

At a jump from i, the new state is chosen according to P. For $i \neq j$, the off–diagonal rates are

$$Q_{ij} = \lambda(i)\,P_{ij},$$

and $Q_{ii} = -\sum_{j\neq i} Q_{ij}$.
Thus

$$Q_{12} = 1 \cdot \tfrac{1}{2} = \tfrac{1}{2}, \quad Q_{13} = 1 \cdot 0 = 0, \quad Q_{11} = -\tfrac{1}{2}.$$

From state 2,

$$Q_{21} = 2 \cdot \tfrac{1}{4} = \tfrac{1}{2}, \quad Q_{23} = 2 \cdot \tfrac{1}{4} = \tfrac{1}{2}, \quad Q_{22} = -\left(\tfrac{1}{2} + \tfrac{1}{2}\right) = -1.$$

From state 3,

$$Q_{31} = 2 \cdot \tfrac{1}{2} = 1, \quad Q_{32} = 2 \cdot 0 = 0, \quad Q_{33} = -1.$$

So

$$Q = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & -1 & 1/2 \\ 1 & 0 & -1 \end{bmatrix}.$$

4. Let X_t be the Q-process taking values in $\{0,1\}$ and set $Q_{1,1} = -\lambda_1 < 0$, $Q_{2,2} = -\lambda_2 < 0$. Express X_t in terms of two independent unit rate Poisson processes $\Pi_1(t)$ and $\Pi_2(t)$.

Solution. If we write

$$X_t = X_0 + \Pi_1\left(\int_0^t \lambda_1(1 - X_s)\, ds\right) - \Pi_2\left(\int_0^t \lambda_2 X_s\, ds\right)$$

then when $X_t = 0$, Π_1 adds 1 at a rate λ_1 and when $X_t = 1$, Π_2 subtracts 1 at a rate λ_2.

Section 2

5. Consider the filtration $\{\mathcal{F}_n\}_{n>0}$ associated with the subset of sequence space $\Omega \subset \{1,2,3\}^{\mathbb{N}}$ whose first three elements are

$$\begin{aligned} \mathcal{F}_0 &= \sigma(C(1)) \\ \mathcal{F}_1 &= \sigma(C(1,2), C(1,3), C(1,1)) \\ \mathcal{F}_2 &= \sigma(C(1,2,1), C(1,2,3), C(1,3,1), C(1,1,2), C(1,1,3), C(1,1,1)) \end{aligned}$$

where C denotes the cylinder sets

$$C(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) = \{\omega \in \Omega : \omega_0 = \varepsilon_0, \ldots, \omega_n = \varepsilon_n\}. \tag{1}$$

Let X_n be an $\{\mathcal{F}_n\}_{n>0}$-adapted time-homogenous Markov chain. Give an example of one possible transition matrix for X_n.

Solution. From $\mathcal{F}_0 = \sigma(C(1))$ we see $X_0 = 1$ almost surely. The atoms of $\mathcal{F}_1$ show that the only possible pairs (X_0, X_1) are $(1,1), (1,2), (1,3)$, so from state 1 we may go to $1,2,3$ but not to any other state. The atoms of $\mathcal{F}_2$ show that the only possible triples (X_0, X_1, X_2) are

$$(1,2,1), (1,2,3), (1,3,1), (1,1,2), (1,1,3), (1,1,1).$$

Thus, from state 2 we may go to 1 or 3 (but not 2), and from state 3 we may go only to 1.

One possible transition matrix consistent with this is

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix},$$

with initial distribution $\mu_0 = (1,0,0)$.

6. Consider the Gaussian process with the kernel

$$K(t,s) = 4\cos(2\pi t)\cos(2\pi s) + \sin(2\pi t)\sin(2\pi s) \tag{2}$$

For $s, t > 0$, compute the conditional distribution

$$\left(\frac{d}{dt}X_t, X_s\right) \mid \{X_0 = 0\};$$

that is, the joint distribution of $(dX_t/dt, X_s)$ conditioned on $X_0 = 0$.

Solution. Write the kernel as

$$K(t,s) = \phi_1(t)\phi_1(s) + \phi_2(t)\phi_2(s),$$

with

$$\phi_1(t) = 2\cos(2\pi t), \qquad \phi_2(t) = \sin(2\pi t).$$

Then we can represent

$$X_t = Z_1\phi_1(t) + Z_2\phi_2(t) = 2Z_1\cos(2\pi t) + Z_2\sin(2\pi t),$$

with Z_1, Z_2 independent $\mathcal{N}(0,1)$.

Differentiate:

$$X_t' = \frac{d}{dt}X_t = -4\pi Z_1\sin(2\pi t) + 2\pi Z_2\cos(2\pi t).$$

At $t = 0$,

$$X_0 = 2Z_1.$$

Conditioning on $X_0 = 0$ forces $Z_1 = 0$, while $Z_2 \sim \mathcal{N}(0,1)$ remains independent. Thus, under $X_0 = 0$,

$$X_t' = 2\pi Z_2\cos(2\pi t), \qquad X_s = Z_2\sin(2\pi s).$$

Hence (X_t', X_s) conditional on $X_0 = 0$ is bivariate normal with mean zero and covariance matrix

$$\mathrm{Cov}\big((X_t', X_s)^\top \mid X_0 = 0\big) = \begin{bmatrix} 4\pi^2\cos^2(2\pi t) & 2\pi\cos(2\pi t)\sin(2\pi s) \\ 2\pi\cos(2\pi t)\sin(2\pi s) & \sin^2(2\pi s) \end{bmatrix}.$$

7. Given an example of a kernel $K(t,s)$ for a mean zero Gaussian process X_t whose realizations have the following properties:

   - X_t is not a constant
   - X_t is differentiable
   - $Y_t = X_{t^2}$ is 2π periodic

Solution. One way to achieve this is to build X_t from a 2π–periodic Gaussian process Z_θ in a smooth way. For example, let Z_θ be a mean–zero Gaussian process on $\mathbb{R}$ with periodic kernel

$$k(\theta,\phi) = \exp\big(-a\big(1 - \cos(\theta - \phi)\big)\big), \qquad a > 0,$$

which is smooth and nonconstant, so its sample paths are differentiable and not constant.

Define $X_t = Z_{\sqrt{t}}$ for $t \geq 0$. Then X_t is mean zero, nonconstant, and differentiable (as a composition of a smooth periodic process with a smooth map $t \mapsto \sqrt{t}$). Its covariance kernel is

$$K(t,s) = k(\sqrt{t}, \sqrt{s}) = \exp\big(-a\big(1 - \cos(\sqrt{t} - \sqrt{s})\big)\big).$$

Now

$$Y_t = X_{t^2} = Z_t,$$

so Y_t has exactly the same law as the 2π–periodic process Z_t. Thus a valid example of such a kernel is

$$K(t,s) = \exp\big(-a\big(1 - \cos(\sqrt{t} - \sqrt{s})\big)\big), \qquad a > 0.$$