Math 106 – Final Practice Problems

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The exam will be given 3/12 and due the last day of finals week. It will cover the material through week 8. You can take it whenever you want during the exam period, but you must give yourself no more than 2 hours (unless you have accommodations) and use no notes, electronics or book.

Markov processes, semigroups, and generators (Chapter 5)

1. Consider the multiplication semigroup $(T_t)_{t\geq 0}$ by $T_t f(x) = e^{tq(x)}f(x)$ where $f, q \in C_0(\mathbb{R})$. Recall that
$$C_0(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous and } \lim_{x\to\pm\infty} f(x) = 0 \right\} \tag{1}$$
and this is a Banach space with norm $||f||_\infty = \sup_x |f(x)|$.

   - Show that T_t is strongly continuous.
   - What is the generator $\mathcal{A}$ of T_t?

   Solution. Since $q \in C_0(\mathbb{R})$, q is bounded and $e^{tq(x)} \to 1$ uniformly as $t \to 0$, so $\|T_t f - f\|_\infty \leq \|e^{tq} - 1\|_\infty \|f\|_\infty \to 0$; hence (T_t) is strongly continuous. The pointwise derivative at $t = 0$ gives $\mathcal{A}f = \lim_{t\downarrow 0}(T_t f - f)/t = q(x)f(x)$ with domain all $f \in C_0(\mathbb{R})$.

2. Consider the Koopman generator
$$(\mathcal{L}f)(x) = x(1-x)f'(x). \tag{2}$$
Find the semigroup $(T_t)_{t\geq 0}$ generated by $\mathcal{L}$, i.e. find an explicit formula for $(T_t f)(x)$.

   Solution. The flow solves $\dot{x} = x(1-x)$ with solution $\phi_t(x) = \dfrac{xe^t}{1 + x(e^t - 1)}$. The Koopman semigroup is composition with the flow: $(T_t f)(x) = f(\phi_t(x))$.

3. Consider the process (X_t, Y_t) on $\mathbb{R} \times \{0, 1\}$ defined by
$$\frac{d}{dt}X_t = Y_t, \tag{3}$$
$$Y_{t+dt} = Y_t + \Pi(ht)(1 - 2Y_t), \quad dt \ll 1 \tag{4}$$
This is an example of a velocity-jump process.

   - Write down the generator $\mathcal{L}$, acting on functions $f : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$.
   - Argue (without formal proof) that $\text{var}(X_t) \to \infty$ as $t \to \infty$.

Solution. Let Π be a Poisson process of rate h. Between jumps Y_t is constant and X_t moves with velocity y. The generator is

$$(\mathcal{L}f)(x,y) = y\,\partial_x f(x,y) + h\big(f(x,1-y) - f(x,y)\big).$$

Because the velocity flips at Poisson times with zero mean but nonzero autocovariance, X_t behaves like an integrated telegraph process whose variance grows linearly in t; hence $\mathrm{var}(X_t) \uparrow \infty$.

4. Consider the process with generator $\mathcal{L}$ acting on some subspace of functions $f : \mathbb{R}\times\{1,2,\cdots,I\} \to \mathbb{R}$ given by

$$\mathcal{L}f(x,i) = D_i\partial_x^2 f(x,i) + \sum_{j=1}^{I} Q_{i,j}(f(x,j) - f(x,i)) \tag{5}$$

where Q is the generator of an irreducible Q-process and $D_i > 0$ for $i = 1,\ldots,I$. Describe the process (X_t, I_t) generated by this operator in words. What will happen to X_t as $t \to \infty$?

Solution. I_t is a continuous-time Markov chain on $\{1,\ldots,I\}$ with rate matrix Q. Conditional on $I_t = i$, the X-coordinate follows a Brownian motion with diffusion coefficient $\sqrt{2D_i}$. Thus (X_t, I_t) is a regime-switching diffusion. Since Q is irreducible, the chain visits each state infinitely often and the marginal X_t diffuses with an effective variance growing like $2(\pi\cdot D)\,t$, where π is the stationary law of Q; consequently $|X_t| \to \infty$ in probability (variance diverges linearly).

5. Consider the generator

$$\mathcal{L}f(x) = \alpha \int_{\mathbb{R}} h(y)\big(f(x-y) - f(x)\big)\,dy \tag{6}$$

where $h(y)$ is a probability density on $\mathbb{R}$ and $\alpha > 0$. Assume there is a well-defined Markov process X_t generated by this linear operator.

(a) Describe the process X_t. Hint: What if you replaced the integral with a sum? Compare to a Q-process.

(b) Describe the filtration $\mathcal{F}_t^X$ in terms of generating sets. Hint: Use the intuition from Gillespie algorithm.

(c) Find the adjoint operator $\mathcal{L}^*$.

Solution. $\mathcal{L}$ is the generator of a compound Poisson process of rate α with i.i.d. jump sizes $Y_k \sim h$; the process makes jumps $X \mapsto X - Y_k$ at Poisson times and is constant between jumps. The natural filtration is generated by the jump times $\{\tau_k \le t\}$ and jump sizes $\{Y_k : \tau_k \le t\}$. For the adjoint, we use the L^2 pairing $\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)\,dx$ and compute $\langle \mathcal{L}f, g\rangle$:

$$\begin{aligned}
\langle \mathcal{L}f, g\rangle &= \alpha \int_{\mathbb{R}}\int_{\mathbb{R}} h(y)\big(f(x-y) - f(x)\big)g(x)\,dy\,dx \\
&= \alpha \int_{\mathbb{R}}\int_{\mathbb{R}} h(y)f(x-y)g(x)\,dy\,dx - \alpha \int_{\mathbb{R}}\int_{\mathbb{R}} h(y)f(x)g(x)\,dy\,dx.
\end{aligned}$$

In the first double integral substitute $z = x - y$ (so $x = z + y$, $dx = dz$):

$$\alpha \int_{\mathbb{R}}\int_{\mathbb{R}} h(y)f(z)g(z+y)\,dy\,dz.$$

The second integral simplifies using $\int h(y)\,dy = 1$:

$$-\alpha \int_{\mathbb{R}} f(x)g(x)\,dx.$$

Combining and renaming $z \to x$:

$$\langle \mathcal{L}f, g\rangle = \int_{\mathbb{R}} f(x)\left[\alpha \int_{\mathbb{R}} h(y)\big(g(x+y) - g(x)\big)\,dy\right] dx = \langle f, \mathcal{L}^* g\rangle,$$

so

$$(\mathcal{L}^* g)(x) = \alpha \int_{\mathbb{R}} h(y)\big(g(x+y) - g(x)\big)\,dy.$$

This corresponds to jumps *upward* by y in the forward (Fokker–Planck) equation, reflecting that the backward generator acts on the starting point while the adjoint acts on the ending point.

Wiener processes, Stochastic integrals and Itô processes (Chapter 6 and 7)

6. Consider the Itô integral

$$I_T = \int_0^T e^t W_t \, dW_t \tag{7}$$

- Compute the variance of I_T
- Compute $\mathbb{E}[I_T \mid \mathcal{F}_s^W]$ for $s < T$.

Solution. By Itô isometry, $\mathrm{Var}(I_T) = \mathbb{E}[I_T^2] = \int_0^T e^{2t}\mathbb{E}[W_t^2]\,dt = \int_0^T te^{2t}\,dt = \frac{1}{2}Te^{2T} - \frac{1}{4}e^{2T} + \frac{1}{4}$. For $s < T$, the increment $\int_s^T e^t W_t\,dW_t$ is independent of $\mathcal{F}_s^W$ with mean 0, so $\mathbb{E}[I_T \mid \mathcal{F}_s^W] = \int_0^s e^t W_t\,dW_t =: I_s$.

7. Evaluate the integral

$$\int_0^T W_t^3 dW_t \tag{8}$$

In particular, express it in terms of non-Itô integrals (the integrands can involve W_t) and W_T.

Solution. Apply Itô to W_t^4: $d(W_t^4) = 4W_t^3\,dW_t + 6W_t^2\,dt$. Hence

$$\int_0^T W_t^3\,dW_t = \tfrac{1}{4}W_T^4 - \tfrac{3}{2}\int_0^T W_t^2\,dt,$$

expressed using only W_T and an ordinary time integral.

8. Compute the iterated integral

$$\int_0^t \int_0^{s_3} \int_0^{s_2} dW_{s_1} dW_{s_2} dW_{s_3} \tag{9}$$

Solution. Let

$$I_t := \int_0^t \int_0^{s_3} \int_0^{s_2} dW_{s_1}\,dW_{s_2}\,dW_{s_3}.$$

First collapse the inner two integrals:

$$J_s := \int_0^s \int_0^u dW_r\, dW_u = \tfrac{1}{2}\left(W_s^2 - s\right)$$

using the Itô isometry (or Itô for W_s^2). Then

$$I_t = \int_0^t J_s\, dW_s = \tfrac{1}{2}\int_0^t (W_s^2 - s)\, dW_s = \tfrac{1}{2}\int_0^t W_s^2\, dW_s - \tfrac{1}{2}\int_0^t s\, dW_s.$$

For the first term, apply Itô to W_s^3:

$$d(W_s^3) = 3W_s^2\, dW_s + 3W_s\, ds \quad \Rightarrow \quad \int_0^t W_s^2\, dW_s = \tfrac{1}{3}W_t^3 - \int_0^t W_s\, ds.$$

For the second term, integrate by parts $d(sW_s) = s\, dW_s + W_s\, ds$ to get

$$\int_0^t s\, dW_s = tW_t - \int_0^t W_s\, ds.$$

Substituting,

$$I_t = \tfrac{1}{2}\left(\tfrac{1}{3}W_t^3 - \int_0^t W_s\, ds\right) - \tfrac{1}{2}\left(tW_t - \int_0^t W_s\, ds\right) = \frac{1}{6}W_t^3 - \frac{1}{2}tW_t = \frac{1}{6}\left(W_t^3 - 3tW_t\right).$$

The bracket is the probabilists' Hermite polynomial $\mathrm{He}_3(x) = x^3 - 3x$, so $I_t = \frac{t^{3/2}}{6}\,\mathrm{He}_3\!\left(\frac{W_t}{\sqrt{t}}\right)$.

9. Let

$$dX_t = a_t dt + b_t dW_t \tag{10}$$
$$dY_t = c_t dt + d_t dW_t \tag{11}$$

Note these processes are driven by the same Wiener process. Let $Z_t = X_t Y_t$ and compute the SDE for Z_t.

Solution. Itô product rule or Ito lemma with $f(x,y) = xy$ yields

$$dZ_t = (a_t Y_t + c_t X_t + b_t d_t)\, dt + (b_t Y_t + d_t X_t)\, dW_t.$$

10. For each of the processes below, rewrite Y_t in the differential form

$$dY_t = u(t,\omega)\, dt + v(t,\omega)\, dB_t$$

for suitable choices of u, v and dimensions.

(a) $Y_t = W_t^2$
(b) $Y_t = 2 + t + e^{W_t}$
(c) $Y_t = W_{1,t}^2 + W_{2,t}^2$ where $W_{1,t}$ and $W_{2,t}$ are independent Wiener processes.

Solution. (a) $dY_t = 2W_t\,dW_t + dt$, so $u = 1$, $v = 2W_t$, one-dimensional.
(b) $dY_t = \big(1 + \frac{1}{2}e^{W_t}\big)dt + e^{W_t}\,dW_t$, so $u = 1 + \frac{1}{2}e^{W_t}$, $v = e^{W_t}$.
(c) $d\mathbf{Y}_t = 2W_{1,t}\,dW_{1,t} + 2W_{2,t}\,dW_{2,t} + 2\,dt$ with two-dimensional Brownian driver; drift $u = 2$, diffusion vector $v = (2W_{1,t}, 2W_{2,t})$.

11. Solve the SDE

$$dX_t = rdt + \alpha X_t dW_t, \quad r, \alpha > 0 \tag{12}$$

Solution. Write the linear SDE in the form $dX_t - \alpha X_t\,dW_t = r\,dt$ and choose the integrating factor

$$M_t := \exp\big(-\alpha W_t + \tfrac{1}{2}\alpha^2 t\big),$$

which satisfies $dM_t = -\alpha M_t\,dW_t + \frac{1}{2}\alpha^2 M_t\,dt$ and cancels the stochastic term in $d(M_t X_t)$. Itô's product rule gives

$$d(M_t X_t) = M_t\,dX_t + X_t\,dM_t + dM_t\,dX_t = M_t(r\,dt) + \big(-\alpha M_t X_t\,dW_t\big) + \alpha M_t X_t\,dW_t = rM_t\,dt.$$

Integrate from 0 to t:

$$M_t X_t = X_0 + r\int_0^t M_s\,ds \quad \Rightarrow \quad X_t = M_t^{-1}\left(X_0 + r\int_0^t M_s\,ds\right).$$

Since $M_t^{-1} = e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$, we obtain the explicit solution

$$X_t = X_0 e^{\alpha W_t - \frac{1}{2}\alpha^2 t} + r\int_0^t e^{\alpha(W_t - W_s) - \frac{1}{2}\alpha^2(t-s)}\,ds,$$

which is the usual variation-of-constants (Duhamel) form for multiplicative noise linear SDEs.

12. Let X_t be an OU process with relaxation rate θ and diffusion coefficient $D = \theta$:

$$dX_t = -\theta X_t\,dt + \sqrt{2\theta}\,dW_t, \qquad X_0 = 0. \tag{13}$$

Define

$$\frac{d}{dt}W_t^\theta = X_t \tag{14}$$

(a) Explain why W_t^θ converges to a Wiener process W_t as $\theta \to \infty$. You don't need to provide a rigorous proof, only an heuristic explanation for why $\lim_{\theta\to 0} W_t^\theta$ has the properties of a Wiener process. Hint: Look at the joint distribution of the increments of W_t^θ.

(b) Let

$$Y_t^\theta = y_0 + \int_0^t Y_s^\theta dW_s^\theta \tag{15}$$

and f be a bounded function. What is the integral equation for $f(Y_t^\theta)$?

(c) Argue based on part (b) that as $\theta \to \infty$ Y_t^θ *does not* converge to the Itô SDE

$$dY_t = Y_t dW_t \tag{16}$$

as we might suspect from part (a)

Solution. (a) Since the variance of an OU process with diffusion coefficient D and relaxation rate θ is D/θ, in this case the variance is 1. As $\theta \to \infty$ the values X_t become uncorrelated and we obtain a process with constant variance and independent values. Integrating this process therefore gives a process with independent increments with constant variance, which is a Wiener process. (b) Because W^θ has C^1 sample paths, we use the ordinary chain rule (no Itô correction):

$$f(Y_t^\theta) = f(y_0) + \int_0^t f'(Y_s^\theta)\, dY_s^\theta = f(y_0) + \int_0^t f'(Y_s^\theta) Y_s^\theta \, dW_s^\theta.$$

(c) Taking the limit $\theta \to \infty$ of the above formula for $f(Y_t^\theta)$ contradicts Ito's lemma.

13. Let X_t solve geometric Brownian motion:

$$dX_t = rX_t\, dt + \sigma X_t\, dW_t, \qquad X_0 = x_0 > 0. \tag{17}$$

Derive the quadratic variation

$$[X, X]_T = \lim_{dt\to 0} \sum_{i=1}^{n} (X_{t_{i+1}} - X_{t_i})^2 \tag{18}$$

Solution. For $dX_t = rX_t\, dt + \sigma X_t\, dW_t$, the quadratic variation is the time integral of the diffusion coefficient squared: $[X, X]_T = \int_0^T \sigma^2 X_t^2\, dt$. Writing $X_t = x_0 \exp\big((r - \frac{1}{2}\sigma^2)t + \sigma W_t\big)$ gives the explicit (random) expression

$$[X, X]_T = \sigma^2 x_0^2 \int_0^T \exp\big(2(r - \tfrac{1}{2}\sigma^2)s + 2\sigma W_s\big)\, ds.$$

Fokker–Planck equation (Chapter 8)

14. Consider the Fokker–Planck equation

$$\begin{aligned} \partial_t p(x, y, t) = -\partial_x\Big(- x(x^2 + y^2 - 1)\, p(x, y, t)\Big) - \partial_y\Big(- y(x^2 + y^2 - 1)\, p(x, y, t)\Big) \\ + D\big(\partial_{xx}p + \partial_{yy}p(x, y, t)\big). \end{aligned} \tag{19}$$

Compute $\lim_{t\to\infty} p(x, y, t)$.

Solution. The drift is gradient of the potential $U(x, y) = \frac{1}{4}\big((x^2 + y^2) - 1\big)^2$ since $\nabla U = (x(r^2 - 1), y(r^2 - 1))$ with $r^2 = x^2 + y^2$. For such gradient systems with constant isotropic diffusion, the invariant density is Gibbs:

$$p_\infty(x, y) = Z^{-1} \exp\left(-\frac{U(x, y)}{D}\right) = Z^{-1} \exp\left(-\frac{(x^2 + y^2 - 1)^2}{4D}\right).$$

Thus $p(x, y, t) \to p_\infty(x, y)$ as $t \to \infty$.

15. Consider geometric Brownian motion (Eq. 17).

(a) Write down the Fokker–Planck equation for the density $p(x, t)$ on $x > 0$.

(b) Show that the lognormal density

$$p(x,t) = \frac{1}{x\,\sigma\sqrt{2\pi t}} \exp\left(-\frac{\left(\log(x/x_0) - (r - \frac{1}{2}\sigma^2)t\right)^2}{2\sigma^2 t} \right)$$

solves that PDE for $t > 0$ (with initial condition $p(x,0) = \delta(x - x_0)$).

Solution. (a) The FP equation is

$$\partial_t p = -\partial_x\big(rx\,p\big) + \tfrac{1}{2}\sigma^2\,\partial_{xx}\big(x^2 p\big), \qquad x > 0.$$

(b) Substituting the given lognormal density (the law of $X_t = x_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$) into the PDE and using that it is the transition density of the SDE verifies it satisfies the equation and the delta initial condition.

Other topics (Importance sampling, Feynman-Kac)

16. Let p and q be two multivariate normal densities with covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$ and means $m \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ respectively.

(a) Derive $L(x)$ such that

$$\mathbb{E}_p[f(X)] = \mathbb{E}_q[f(X)L(X)] \tag{20}$$

(b) Discuss the relationship between the formula you obtained for $L(x)$ and the Radon–Nikodym derivative in Girsanov's theorem. In particular, state how the terms in Girsanov's theorem are related to the terms in your formula.

Solution. Since the covariances coincide, the likelihood ratio is $L(x) = p(x)/q(x)$. Expanding the exponents explicitly:

$$\begin{aligned}\log L(x) &= -\tfrac{1}{2}(x-m)^\top\Sigma^{-1}(x-m) + \tfrac{1}{2}(x-v)^\top\Sigma^{-1}(x-v)\\ &= (m-v)^\top\Sigma^{-1}x - \tfrac{1}{2}\Big(m^\top\Sigma^{-1}m - v^\top\Sigma^{-1}v\Big).\end{aligned}$$

Writing $\delta = m - v$ and completing in terms of the reference mean v:

$$\log L(x) = \underbrace{\delta^\top\Sigma^{-1}(x-v)}_{\text{(I)}} - \underbrace{\tfrac{1}{2}\,\delta^\top\Sigma^{-1}\delta}_{\text{(II)}},$$

so $L(x) = \exp\big(\text{(I)} - \text{(II)}\big)$.

Connection to Girsanov. Girsanov's theorem says that if W_t is a standard Brownian motion under $\mathbb{Q}$ and we add a drift θ_s, the Radon–Nikodym derivative for the change of measure to $\mathbb{P}$ is

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = \exp\left(\underbrace{\int_0^T \theta_s\,dW_s}_{\text{(I}')} - \underbrace{\tfrac{1}{2}\int_0^T \theta_s^2\,ds}_{\text{(II}')} \right).$$

The two formulas are structurally identical, with the following term-by-term dictionary:

	Gaussian (finite-dim.)	Girsanov (path space)
	$\delta^\top \Sigma^{-1}(x-v)$	$\int_0^T \theta_s \, dW_s$
	$\frac{1}{2}\,\delta^\top \Sigma^{-1}\delta$	$\frac{1}{2}\int_0^T \theta_s^2 \, ds$

17. Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. Consider the Ornstein–Uhlenbeck process

$$dX_t = -X_t dt + \frac{1}{2}\, dW_t, \qquad X_0 = 0. \tag{1}$$

and suppose we wish to estimate

$$\mathbb{E}[e^{\int_0^t X_s ds}] \tag{21}$$

from Monte Carlo simulations via the estimator

$$\hat{I}_m = \frac{1}{m}\sum_{i=1}^{m} e^{\int_0^t X_{i,s} ds} \tag{22}$$

where $\{X_{i,t}\}_{s\in[0,t],i=1,\ldots,m}$ are independent realizations of X_t. Find a lower bound on m such that the estimator will have a mean-squared error of less than 0.1. Hint: $\int_0^t X_s ds$ is normal, so $Z_t = e^{\int_0^t X_s ds}$ is lognormal.

Solution. Let $I_t = \int_0^t X_s ds$. For the OU in (**??**) with $\kappa = 1$ and $\sigma = \frac{1}{2}$, $\mathrm{Var}(I_t) = \frac{1}{8}\big(2t - 3 + 4e^{-t} - e^{-2t}\big)$ (from integrating the covariance). Then $Z_t = e^{I_t}$ is lognormal with $\mathbb{E}[Z_t] = e^{\frac{1}{2}\mathrm{Var}(I_t)}$ and $\mathrm{Var}(Z_t) = e^{\mathrm{Var}(I_t)}\big(e^{\mathrm{Var}(I_t)} - 1\big)$. The sample mean is unbiased, so $\mathrm{MSE}(\hat{I}_m) = \mathrm{Var}(Z_t)/m$. Require

$$m \geq \frac{e^{\mathrm{Var}(I_t)}\big(e^{\mathrm{Var}(I_t)} - 1\big)}{0.1}.$$

For example, at $t = 1$, $\mathrm{Var}(I_1) \approx 0.0420$, giving $\mathrm{Var}(Z_1) \approx 0.043$ and $m \gtrsim 1$; at $t = 5$, $\mathrm{Var}(I_5) \approx 0.878$, so $\mathrm{Var}(Z_5) \approx 1.55$ and $m \gtrsim 16$.