Examples

1) Let $Y \sim \text{Bernoulli}(1/4)$

$Z = 4Y + 1$

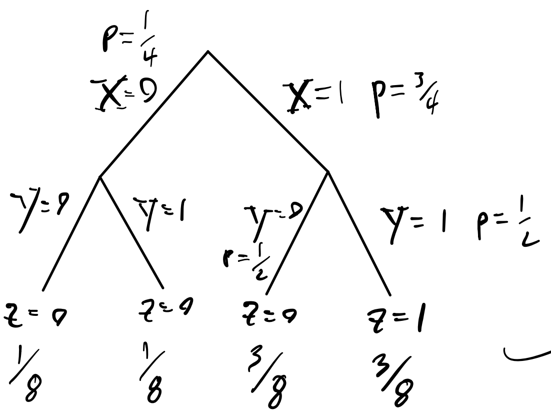
Q: write down sample space and probability distribution

$S = \{1, 5\}$, $P(Z=1) = 3/4$

$P(Z=5) = 1/4$ $\square$

2) Let $Y \sim \text{Bernoulli}(1/2)$, $X \sim \text{Bernoulli}(3/4)$

$Z = XY$ Q: what is distribution of Z?



$P(Z=1) = 3/8$

$P(Z=0) = \frac{1}{8} + \frac{1}{8} + 3/8 = 5/8$

$\Rightarrow Z \sim \text{Bernoulli}(3/8)$ $\square$

Application: what is chance machine breaks if both components X and Y need to fail?

Can you think of other applications?

Example 2.3.4 (Geometric Distribution)

We flip a coin until we get heads

let Y = # of coin flips

In other words,

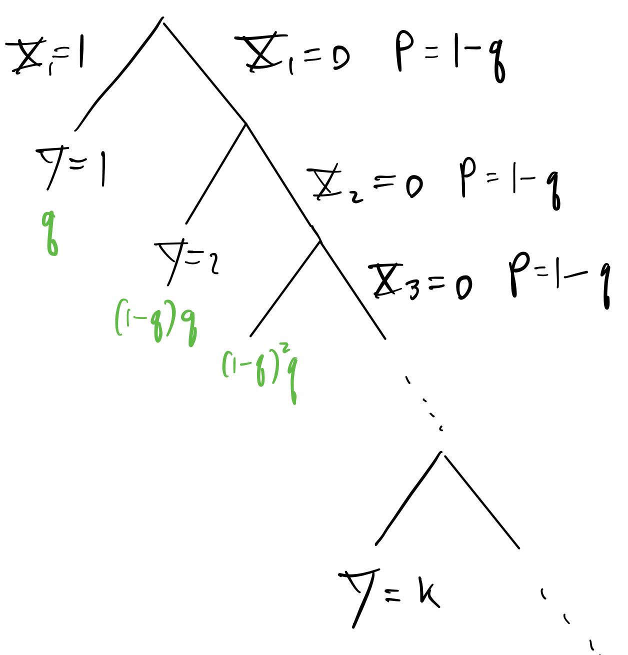
$$X_i \sim \text{Bernoulli}(q) \qquad i = 1, 2, \dots$$

$$Y = \min_{i \geq 0} \{ i : X_i = 1 \}$$

We call Y a geometric r.v. and write

$$Y \sim \text{Geometric}(q)$$

Sample space $S = \{1, 2, 3, \dots, \infty\}$



$$P(\{Y = k\}) = (1-q)^{k-1} q$$

Conditional Probability and independence (ER 1.5)

Motivating Question:

How do probabilities change when we learn new information?

Example

Let $X = \begin{cases} 1 & \text{if patient responds to chemo treatment} \\ 0 & \text{if not} \end{cases}$

Model: $X \sim \text{Bernoulli}(q)$

$$\Rightarrow P(X=1) = \frac{\#\{\text{respond to treatment}\}}{\#\{\text{total}\}}$$

Additional info: patient has gene A

(Such as IKZF1 relevant for Lymphoblastic Leukemia)

Q: what is new prob?

$$P(X=1 \mid \text{have gene } A) = \frac{\#\{\text{respond to treatment AND have gene } A\}}{\#\{\text{have gene } A\}}$$

"given that"

Or if we let $Y = \begin{cases} 1 & \text{has gene } A \\ 0 & \text{otherwise} \end{cases}$

We can rewrite the probability above as

$$P(X=1 \mid Y=1) = \frac{\#\{X=1 \cap Y=1\}}{\#\{Y=1\}}$$

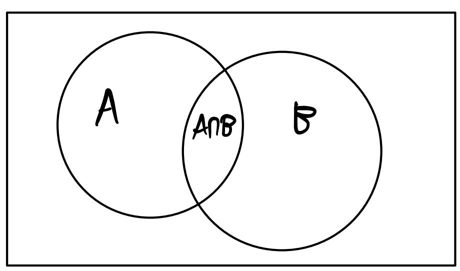
$$= \frac{\#\{X=1 \cap Y=1\} / \#\{\text{total samples}\}}{\#\{Y=1\} / \#\{\text{total samples}\}}$$

$$= \frac{P(\{X=1\} \cap \{Y=1\})}{P(Y=1)} \qquad \square$$

Conditional probability (Def 1.5.1/2.8.1)

for events $A, B \in E$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$



for random variables

$$P(X=x \mid Y=y) = \frac{P(X=x \cap Y=y)}{P(Y=y)}$$

Notice if $P(X=x \cap Y=y) = P(X=x)P(Y=y)$

then

$$P(X=x \mid Y=y) = \frac{\cancel{P(Y=y)}}{\cancel{P(Y=y)}} P(X=x)$$

Independence (Definition 1.5.2 in EK)

Events $A, B \in E$ are independent

if

$$P(A \cap B) = P(A)P(B)$$

R.v. X and Y are independent

if

$$P(\{X=x\} \cap \{Y=y\}) = P(X=x)P(Y=y)$$

Note: textbook treats concepts separately for events vs. r.v.

for r.v. we will write

$$P(\{X=x\} \cap \{Y=y\}) = P(X=x, Y=y)$$

which we call joint distribution (Def 2.7.3 in EK)

also see 2.7.1

$P(X=x)$ is <u>marginal distribution</u> (thm 2.7.4 in EK)

By property #4 of probability distribution, we have

$$P(X=x) = \sum_{y} P(X=x, Y=y)$$

<u>Example</u> Y_A = mutation on gene A

Y_B = mutation on gene B

$$P(Y_A = y_A, Y_B = y_B) = \begin{cases} 1/2 & \text{if } y_A = 0, y_B = 0 \\ 1/8 & \text{if } y_A = 0, y_B = 1 \\ 1/8 & \text{if } y_A = 1, y_B = 0 \\ 1/4 & \text{if } y_A = 1, y_B = 1 \end{cases}$$

another way to write this:

Y_A \ Y_B	0	1
0	1/2	1/8
1	1/8	1/4

<u>Q: what is marginal distribution of Y_A?</u>

$$P(Y_A=1) = P(Y_A=1, Y_B=0) + P(Y_A=1, Y_B=1)$$

$$= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

$$\Rightarrow P(Y_A=0) = \frac{5}{8} \Rightarrow Y_A \sim \text{Bernoulli}(3/8)$$

Similarly $Y_B \sim \text{Bernoulli}(3/8)$ $\square$

$$\begin{array}{cc|cc|c} & & Y_B & & \\ & & 0 & 1 & \\ \hline Y_A & 0 & 1/2 & 1/8 & 5/8 \\ & 1 & 1/8 & 1/4 & 3/8 \\ \hline & & 5/8 & 3/8 & \end{array}$$

Q: Are Y_A and Y_B independent? what are conditional probabilities?

$$P(Y_A=1)P(Y_B=1) = \frac{9}{64}$$

$$\neq P(\{Y_A=1\} \cap \{Y_B=1\}) = \frac{1}{4}$$

Not independent!

$$P(Y_A = 1 \mid Y_B = 0) = \frac{P(Y_A = 1, Y_B = 0)}{P(Y_B = 0)}$$

$$= {}^{1}/_{8} \Big/ {}^{5}/_{8} = \frac{1}{5}$$

$\square$

Review Example 2.7.1, 2.7.5/2.8.1

<u>Other properties of conditional probabilities:</u>

Note that $P(A \cap B) = P(B|A)P(A)$

<u>Theorem 1.5.1</u>

Let $A_1, A_2, \ldots$ be event s.t

$$S = A_1 \cup A_2 \ldots$$

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots$$

$\boxed{\text{End of 9/18 material}}$

Example (2 revisited)

Let $Y \sim \text{Bernoulli}(1/2)$, $X \sim \text{Bernoulli}(3/4)$, $Z = XY$

Before we saw that $Z \sim \text{Bernoulli}(3/8)$

we can interpret this as marginal distribution of Z

Q what is joint distribution $P(Z=z, X=x, Y=y)$?

$$P(Z=0, X=0, Y=0) = P(Z=0 \mid X=0, Y=0) \times P(X=0, Y=0)$$

$$= 1 \times 3/8$$

$$P(Z=0, X=x, Y=y) = P(X=x, Y=y)$$

if $x=0$ or $y=0$

otherwise $= 0$

$$P(Z=1) = \sum_{x,y} P(Z=1 \mid X=x, Y=y)\, P(X=x, Y=y)$$

$$= P(X=1, Y=1)$$

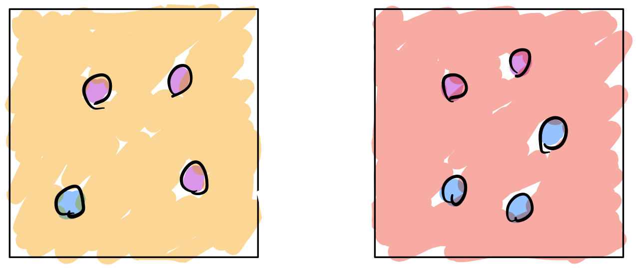
<u>Bayes' theorem</u> (thm 1.5.2)

A, B events w/ $P(B) > 0$,

$$P(A \mid B) = \frac{P(A)}{P(B)} P(B \mid A)$$

Example

Box A　　　　Box B



Randomly select box and pick random ball. If you get blue, what is the chance you selected Box B?

$$P(B \mid \text{blue}) = \frac{P(\text{blue} \mid B)\,P(B)}{P(\text{blue})}$$

$$P(\text{blue} \mid B) = 3/5, \quad P(B) = \frac{1}{2}$$

By Thm 1.5-1,

$$P(\text{blue}) = P(\text{blue} \mid B)P(B) + P(\text{blue} \mid A)P(A)$$

$$= 3/5 \cdot 1/2 + \frac{1}{4}\cdot\frac{1}{2} = \frac{3}{10} + \frac{1}{8}$$

$$\Rightarrow P(B \mid \text{blue}) = 3/5 \cdot 1/2 \Big/ \left(3/10 + 1/8\right)$$

See Example 1.5.3